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New Inertial Azimuth Finder Apparatus

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A new device for finding the azimuth of a rigid body is described. The device is based on the reading of a rotating, vertically placed accelerometer. The functional relationship between the measured specific force and the azimuth information is derived for leveled and tilted apparatus when the latter is at rest or when it is in a straight and level motion. A robust algorithm for extracting the azimuth from the accelerometer measurements is presented, an experimental setup is described, and test results are shown and analyzed. It is shown that the new device is accurate, has low sensitivity to common error sources, and yields the azimuth very fast. The device can be used for azimuth finding in terrestrial missions as well as in orbiting spacecraft.

Nomenclature

A	=	auxiliary computed expression or specific force amplitude
a	=	point on the platform where the accelerometer is located
a'	=	general gravitational force
$\boldsymbol{a}$	=	acceleration vector
$\boldsymbol{a}_a$	=	acceleration vector sensed at point a
$\boldsymbol{a}_o$	=	acceleration vector sensed at point o
$\boldsymbol{a}_r$	=	acceleration vector of point a with respect to point o
B	=	auxiliary computed expression
b	=	body (vehicle) coordinates
$\boldsymbol{b}$	=	acceleration vector
C	=	auxiliary computed expression
$\boldsymbol{c}$	=	general vector
$\boldsymbol{c}_{D_b}$	=	D_b component of $\boldsymbol{c}_b$
$\boldsymbol{c}_q$	=	column vector consisting of the components of the general vector $\boldsymbol{c}$ expressed in the general q coordinates
$\boldsymbol{c}_q^T$	=	transpose of $\boldsymbol{c}_q$
$\boldsymbol{D}$	=	down axis of the l coordinates
D_b^l	=	transformation matrix from the l to the b coordinate system
$\boldsymbol{D}_q$	=	down axis of the general q coordinates
d_{ij}	=	i, j element of D_b^l
$\mathrm{d}\boldsymbol{c}/\mathrm{d}t\|_q$	=	time derivative of the general vector $\boldsymbol{c}$ as seen by an observer in the general q coordinates
$\boldsymbol{E}$	=	east axis of the l coordinates or auxiliary computed expression
$\boldsymbol{E}_q$	=	east axis of the general q coordinates
e	=	Earth coordinates
F	=	auxiliary computed expression
f_D	=	down component of the specific force sensed at point a
G	=	auxiliary computed expression
g	=	magnitude of $\boldsymbol{g}$
$\boldsymbol{g}$	=	gravity vector
H	=	heading
H_c	=	computed, measured, or assumed H
h	=	height
i	=	inertial coordinates
K	=	computed value
L	=	latitude
l	=	local-level north pointing coordinates
M	=	auxiliary computed expression
$\boldsymbol{m}$	=	general vector
$\boldsymbol{N}$	=	north axis of the l coordinates or auxiliary computed expression
$\boldsymbol{N}_q$	=	north axis of the general q coordinates
$\boldsymbol{n}$	=	general vector
O	=	origin of apparatus coordinates
P	=	auxiliary computed expression
p	=	platform coordinates
Q	=	auxiliary computed expression
q	=	general coordinates
R_E	=	radius of curvature for east motion on Earth
R_N	=	radius of curvature for north motion on Earth
r	=	magnitude of $\boldsymbol{r}$
$\boldsymbol{r}$	=	radius vector from the origin of coordinates to point a on the platform
t	=	time
U	=	auxiliary computed expression
V	=	auxiliary computed expression or velocity
V_E	=	east component of velocity on Earth
V_N	=	north component of velocity on Earth
W	=	auxiliary computed expression
z_b	=	z axis of the b coordinates
α	=	angle between platform north and body north
ζ	=	inertial azimuth finding apparatus (IAFA) indicated phase
η	=	angle between $\boldsymbol{N}$ and $\boldsymbol{r}$
θ	=	body pitch angle
μ	=	computed angle
ξ	=	IAFA indicated phase
$\boldsymbol{\rho}$	=	angular velocity of the l with respect to the e coordinates
ϕ	=	body roll angle
ψ	=	body yaw angle
Ω	=	magnitude of $\boldsymbol{\Omega}$
$\boldsymbol{\Omega}$	=	Earth angular velocity with respect to the i coordinates
Ω_H	=	horizontal component of Earth rate, $\sqrt{(\Omega_{N_b}^2 + \Omega_{E_b}^2)}$
Ω_0	=	magnitude of $\boldsymbol{\Omega}_0$
$\boldsymbol{\Omega}_0$	=	rotation rate of the platform with respect to the b coordinates
$\boldsymbol{\omega}$	=	angular velocity of the l with respect to the i coordinates
$\boldsymbol{\omega}^{q-u}$	=	angular velocity of the general q system with respect to the general u coordinates
$\boldsymbol{\omega}^*$	=	angular velocity of the b with respect to the l coordinates

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I. Introduction

ONE of the most troublesome problems in navigation and positioning is the determination of the azimuth, particularly the initial azimuth of a vehicle. There are many cases where inaccurate, and thus inexpensive, inertial sensors are sufficient for the navigation task itself, but more accurate sensors are needed only to satisfy the accuracy requirements of the initial azimuth determination. In inertial navigation systems, for example, the azimuth is normally determined by gyrocompassing, which relies on the equivalent east gyro. The constant drift rate of that gyro sets the lower limit of the attainable azimuth determination accuracy. Therefore, there has been a constant effort to find new gyroscopic devices that, on one hand, are accurate enough and, on the other hand, are inexpensive. The second feature of gyrocompassing is the relative long duration needed to complete the process. This is in contrast to the leveling process, needed for complete system initial alignment, which is faster and more accurate. The difference between the process of initial azimuth alignment and that of leveling stems from the difference between the signals used to accomplish the two tasks. As is well known, the measurement of two, noncollinear vectors, in the reference and in the body coordinates, are sufficient for attitude determination. Fortunately, Earth has two such vectors, namely, the gravity vector and the Earth rate vector. However, the intensity of gravity is more than 10^5 larger than that of the Earth rate. Therefore, leveling, which is accomplished by accelerometers that measure the gravity vector, is an easier task to perform than azimuth determination, which is accomplished by gyroscopes that measure Earth rate. Attempts to amplify the earth rate signal electronically are doomed to failure because this also increases the noise level.

Recently a new inertial azimuth finding apparatus (IAFA) was invented[1] that overcomes the deficiencies of gyrocompassing; namely, it is an accelerometer rather than gyro-based device, it amplifies a phenomenon related to the Earth-rate vector, and it is insensitive to sensor bias or scale factor errors. The phenomenon is that of Coriolis acceleration, which is sensed at a point moving in a rotating coordinate system. It is well known that when a point mass is moving at a velocity $\boldsymbol{V}$ with respect to a coordinate system that rotates at an angular velocity $\boldsymbol{\omega}$, then that point experiences a Coriolis acceleration[2] whose value is $2\boldsymbol{\omega} \times \boldsymbol{V}$. Suppose that the point rotates at the rate of turn $\boldsymbol{\Omega}_0$ on a circle whose radius is r. Then obviously, the translatory velocity $\boldsymbol{V}$ is equal to $\boldsymbol{\Omega}_0 \times \boldsymbol{r}$. Consequently, the Coriolis acceleration is $2\boldsymbol{\omega} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r})$. For simplicity of explanation, consider the case where the rotating point is placed on the equator and the plane of rotation is tangential to the Earth surface. An inspection of Fig. 1 reveals that when $\eta = 0$ the rotating point a is on the Earth rate vector $\boldsymbol{\Omega}$, and $\boldsymbol{V}$ points east. In this case, the Coriolis acceleration points down and its value is $2\Omega\Omega_0 r$. It is easy to see that when $\eta = 90^\circ$, then $\boldsymbol{\Omega}$ and $\boldsymbol{V}$ are collinear and thus the Coriolis acceleration is zero. Similarly, when $\eta = 180^\circ$, then $\boldsymbol{V}$ is again perpendicular to $\boldsymbol{\Omega}$, but now it points west so that the Coriolis acceleration points up and its down axis value is $-2\Omega\Omega_0 r$. In conclusion, when we examine the Coriolis acceleration, we realize that it is a periodic signal that takes its maximal value when the point a crosses the north direction, takes its minimal value when it crosses the south direction, and crosses zero when a crosses either the east or the west direction. We realize then that if we read this Coriolis acceleration we can detect north. If we move the rotating point to another latitude, then the situation changes somewhat, but as long as $\boldsymbol{\Omega}$ has a component in the direction of the local north, we still observe this kind of Coriolis acceleration. In other words, we can discover the north direction if the rotating point is not placed at or in the vicinity of the poles. Note that we can control the amplitude of the Coriolis acceleration by changing either Ω_0, the value of the rotation rate of the point, and/or the radius r of the rotation circle.

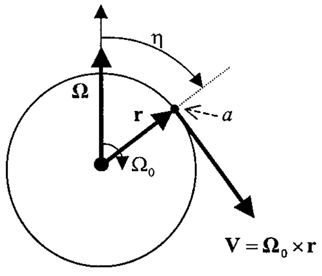

Fig. 1 Vectors involved in the Coriolis acceleration generation at a rotating point on the equator.

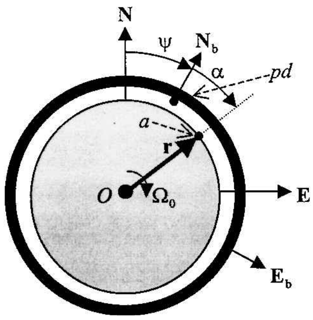

Fig. 2 Overview of the inertial azimuth finding apparatus with an azimuth misalignment only.

In the next section we describe the operation of the device in a more technical fashion. In Sec. III, we develop the general acceleration equation that the apparatus accelerometer senses. In Sec. IV, we develop the equations that connect the accelerometer reading with the azimuth, both in the leveled and tilted IAFA cases when the latter is at rest. In Sec. V, we do the same for the case where the IAFA is in a straight and level motion. In Sec. VI, we present a robust algorithm for extracting the azimuth information from the accelerometer reading. In Sec. VII, we describe the results that were obtained when testing the concept using a breadboard model of the IAFA. Finally, in Sec. VIII, we present the conclusion of this work.

II. Apparatus Operation

Figure 2 presents a top view of the apparatus. The outside ring represents the vehicle body. The inner circular plate rotates about the center point o at the rate Ω_0. The rotation axis D_D is the down axis of the body coordinates. An accelerometer is mounted at point a on the circumference of the plate, at a distance r from the center, and its sensitive axis is mounted downward in the direction of D_D. The accelerometer is mounted along N_p, the north axis of the platform system (not shown in Fig. 2). At time $t = 0$, N_p and N_b coincide; therefore, $\alpha = \Omega_0 t$. Figure 2 shows the case where the rotating plate is tangential to the Earth surface; consequently, D_D is identical, in this case, to D, the down axis of the l coordinate system. However, N_b, the body north axis, is pointing at an azimuth angle ψ, away from N, the local north. In other words, the body system is leveled but misaligned in azimuth by the angle ψ. The objective of IAFA is to determine the value of ψ.

A photodetector is mounted on the body north axis at point pd, where it records the time when the accelerometer passes by. On the other hand, as was shown in Sec. I and will be shown in Sec. IV, the accelerometer output indicates the time when the accelerometer is aligned with the local north direction. In principle, the time difference between these two events can be easily translated to the azimuth misalignment angle ψ. However, as will be shown in Sec. VI, a more robust method can be used for extracting ψ from the accelerometer signal.

Next we show the functional relationship between the accelerometer output and ψ. To this, we first compute the specific force sensed by the accelerometer in general, and then we reduce it to special cases of a resting vehicle in both leveled and unleveled conditions and to a vehicle moving at a constant velocity. (Note that accelerometers sense specific force and not acceleration.)

III. General Acceleration Equation

The specific force at point a of Fig. 1 is the sum of the gravity acceleration $\boldsymbol{g}$ and the translatory acceleration (Ref. 2, page 18), that is,

$$\boldsymbol{f} = \boldsymbol{a} - \boldsymbol{g} \tag{1}$$

The translatory acceleration can be expressed as the sum of two components. One component is $\boldsymbol{a}_o$, the acceleration vector sensed at point o, and the other one is $\boldsymbol{a}_r$, the acceleration of point a with respect to point o; that is,

$$\boldsymbol{a}_a = \boldsymbol{a}_o + \boldsymbol{a}_r \tag{2}$$

Normally $\boldsymbol{a}_o$ is measured by an inertial measurement unit (IMU) that is installed on the vehicle. Then to find the acceleration measured by the apparatus accelerometer, we need to compute $\boldsymbol{a}_r$. Since

$$\boldsymbol{a}_r = \left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_i \tag{3}$$

we have to compute $\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2|_i$. To meet this end write[3]

$$\left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_i = \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_p + \boldsymbol{\omega}^{p-i} \times \boldsymbol{r} \tag{4}$$

but $\boldsymbol{r}$ is constant in the platform coordinates p; therefore,

$$\left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_p = 0 \tag{5}$$

Consequently, Eq. (4) becomes

$$\left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_i = \boldsymbol{\omega}^{p-i} \times \boldsymbol{r} \tag{6}$$

It is evident that

$$\boldsymbol{\omega}^{p-i} = \boldsymbol{\Omega}_0 + \boldsymbol{\omega}^* + \boldsymbol{\Omega} + \boldsymbol{\rho} \tag{7}$$

then substitution of Eq. (7) into Eq. (6) yields

$$\left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_i = \boldsymbol{\Omega}_0 \times \boldsymbol{r} + \boldsymbol{\omega}^* \times \boldsymbol{r} + \boldsymbol{\Omega} \times \boldsymbol{r} + \boldsymbol{\rho} \times \boldsymbol{r} \tag{8}$$

A second differentiation of Eq. (6) yields

$$\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_i = \left.\frac{\mathrm{d}\boldsymbol{\omega}^{p-i}}{\mathrm{d}t}\right|_i \times \boldsymbol{r} + \boldsymbol{\omega}^{p-i} \times \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_i \tag{9}$$

Now

$$\left.\frac{\mathrm{d}\boldsymbol{\omega}^{p-i}}{\mathrm{d}t}\right|_i = \left.\frac{\mathrm{d}\boldsymbol{\omega}^{p-i}}{\mathrm{d}t}\right|_p + \boldsymbol{\omega}^{p-i} \times \boldsymbol{\omega}^{p-i} \tag{10}$$

but $\boldsymbol{\omega}^{p-i} \times \boldsymbol{\omega}^{p-i} = 0$, therefore,

$$\left.\frac{\mathrm{d}\boldsymbol{\omega}^{p-i}}{\mathrm{d}t}\right|_i = \left.\frac{\mathrm{d}\boldsymbol{\omega}^{p-i}}{\mathrm{d}t}\right|_p \tag{11}$$

Thus, Eq. (9) can be written as

$$\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_i = \left.\frac{\mathrm{d}\boldsymbol{\omega}^{p-i}}{\mathrm{d}t}\right|_p \times \boldsymbol{r} + \boldsymbol{\omega}^{p-i} \times \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_i \tag{12}$$

Substitution of Eqs. (7) and (8) into the last equation yields

$$\begin{aligned}\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_i &= \left(\left.\frac{\mathrm{d}\boldsymbol{\Omega}_0}{\mathrm{d}t}\right|_p + \left.\frac{\mathrm{d}\boldsymbol{\omega}^*}{\mathrm{d}t}\right|_p + \left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_p + \left.\frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t}\right|_p\right) \times \boldsymbol{r} + \big(\boldsymbol{\Omega}_0 + \boldsymbol{\omega}^* \\ &\quad + \boldsymbol{\Omega} + \boldsymbol{\rho}\big) \times \big(\boldsymbol{\Omega}_0 \times \boldsymbol{r} + \boldsymbol{\omega}^* \times \boldsymbol{r} + \boldsymbol{\Omega} \times \boldsymbol{r} + \boldsymbol{\rho} \times \boldsymbol{r}\big)\end{aligned} \tag{13}$$

Consequently,

$$\begin{aligned}\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_i &= \left.\frac{\mathrm{d}\boldsymbol{\Omega}_0}{\mathrm{d}t}\right|_p \times \boldsymbol{r} + \left.\frac{\mathrm{d}\boldsymbol{\omega}^*}{\mathrm{d}t}\right|_p \times \boldsymbol{r} + \left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_p \times \boldsymbol{r} + \left.\frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t}\right|_p \times \boldsymbol{r} \\ &\quad + \boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) + \boldsymbol{\Omega}_0 \times (\boldsymbol{\omega}^* \times \boldsymbol{r}) + \boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \\ &\quad + \boldsymbol{\Omega}_0 \times (\boldsymbol{\rho} \times \boldsymbol{r}) + \boldsymbol{\omega}^* \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) + \boldsymbol{\omega}^* \times (\boldsymbol{\omega}^* \times \boldsymbol{r}) \\ &\quad + \boldsymbol{\omega}^* \times (\boldsymbol{\Omega} \times \boldsymbol{r}) + \boldsymbol{\omega}^* \times (\boldsymbol{\rho} \times \boldsymbol{r}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) \\ &\quad + \boldsymbol{\Omega} \times (\boldsymbol{\omega}^* \times \boldsymbol{r}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) + \boldsymbol{\Omega} \times (\boldsymbol{\rho} \times \boldsymbol{r}) \\ &\quad + \boldsymbol{\rho} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) + \boldsymbol{\rho} \times (\boldsymbol{\omega}^* \times \boldsymbol{r}) \\ &\quad + \boldsymbol{\rho} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) + \boldsymbol{\rho} \times (\boldsymbol{\rho} \times \boldsymbol{r})\end{aligned} \tag{14}$$

IV. Azimuth Finding at Rest

Nominally, $\boldsymbol{\Omega}_0$ is constant in the p system; therefore,

$$\left.\frac{\mathrm{d}\boldsymbol{\Omega}_0}{\mathrm{d}t}\right|_p = 0 \tag{15}$$

The body is not rotating with respect to l; therefore,

$$\boldsymbol{\omega}^* = 0 \tag{16}$$

nor does the body move with respect to the ground; that is, l is not moving with respect to e, therefore,

$$\boldsymbol{\rho} = 0 \tag{17}$$

Using Eqs. (15–17) in Eq. (14) yields

$$\begin{aligned}\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_i &= \left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_p \times \boldsymbol{r} + \boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) + \boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \\ &\quad + \boldsymbol{\Omega} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})\end{aligned} \tag{18}$$

Let us now examine each term on the right-hand side of Eq. (18). We can formally write

$$\left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_l = \left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_p + \left(\boldsymbol{\Omega}_0 + \boldsymbol{\omega}^*\right) \times \boldsymbol{\Omega} \tag{19}$$

Because the vehicle is not moving, $\boldsymbol{\Omega}$ is constant in l; therefore,

$$\left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_l = 0 \tag{20}$$

When Eqs. (16) and (20) are used, Eq. (19) becomes

$$\left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_p = -\boldsymbol{\Omega}_0 \times \boldsymbol{\Omega} \tag{21}$$

From the last equation it is clear that the first term on the right-hand side of Eq. (18) can be written as follows:

$$\left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_p \times \boldsymbol{r} = -(\boldsymbol{\Omega}_0 \times \boldsymbol{\Omega}) \times \boldsymbol{r} = \boldsymbol{r} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{\Omega}) \tag{22}$$

and so Eq. (18) can be written as

$$\begin{aligned}\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_i &= \boldsymbol{r} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{\Omega}) + \boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) + \boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \\ &\quad + \boldsymbol{\Omega} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})\end{aligned} \tag{23}$$

We use the well-known identity[4]

$$\boldsymbol{c} \times (\boldsymbol{m} \times \boldsymbol{n}) = \boldsymbol{m}(\boldsymbol{c} \cdot \boldsymbol{n}) - \boldsymbol{n}(\boldsymbol{c} \cdot \boldsymbol{m}) \tag{24}$$

to write

$$\boldsymbol{r} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{\Omega}) = \boldsymbol{\Omega}_0(\boldsymbol{r} \cdot \boldsymbol{\Omega}) - \boldsymbol{\Omega}(\boldsymbol{r} \cdot \boldsymbol{\Omega}_0) \tag{25a}$$

$$\boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) = \boldsymbol{\Omega}_0(\boldsymbol{\Omega}_0 \cdot \boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{\Omega}_0 \cdot \boldsymbol{\Omega}_0) \tag{25b}$$

$$\boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{\Omega}(\boldsymbol{\Omega}_0 \cdot \boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{\Omega}_0 \cdot \boldsymbol{\Omega}) \tag{25c}$$

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) = \boldsymbol{\Omega}_0(\boldsymbol{\Omega} \cdot \boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}_0) \tag{25d}$$

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{\Omega}(\boldsymbol{\Omega} \cdot \boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}) \tag{25e}$$

Because $\boldsymbol{r} \perp \boldsymbol{\Omega}_0$, then, $\boldsymbol{r} \cdot \boldsymbol{\Omega}_0 = 0$ and, therefore, Eqs. (25a–25c) become

$$\boldsymbol{r} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{\Omega}) = \boldsymbol{\Omega}_0(\boldsymbol{r} \cdot \boldsymbol{\Omega}) \tag{26a}$$

$$\boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) = -\boldsymbol{r}(\boldsymbol{\Omega}_0 \cdot \boldsymbol{\Omega}_0) \tag{26b}$$

$$\boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = -\boldsymbol{r}(\boldsymbol{\Omega}_0 \cdot \boldsymbol{\Omega}) \tag{26c}$$

Substitution of Eqs. (25d) and (25e) and (26a–26c) into Eq. (23) yields

$$\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_i = \boldsymbol{\Omega}_0(\boldsymbol{r}\cdot\boldsymbol{\Omega}) - \boldsymbol{r}(\boldsymbol{\Omega}_0\cdot\boldsymbol{\Omega}_0) - \boldsymbol{r}(\boldsymbol{\Omega}_0\cdot\boldsymbol{\Omega}) + \boldsymbol{\Omega}_0(\boldsymbol{\Omega}\cdot\boldsymbol{r})$$
$$- \boldsymbol{r}(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_0) + \boldsymbol{\Omega}(\boldsymbol{\Omega}\cdot\boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}) \tag{27}$$

The specific force $\boldsymbol{f}_{D_b}$ contains only the body down components of $\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2|_i$. Because $\boldsymbol{r}$ is perpendicular to the direction of $\boldsymbol{f}_{D_b}$, all components of $\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2|_i$ along $\boldsymbol{r}$ are not present in the down direction of $\mathrm{d}^2\boldsymbol{r}/\mathrm{d}t^2|_i$. Therefore, from Eq. (27) we obtain

$$\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_{iD_b} = \boldsymbol{\Omega}_{0D_b}(\boldsymbol{r}\cdot\boldsymbol{\Omega}) + \boldsymbol{\Omega}_{0D_b}(\boldsymbol{\Omega}\cdot\boldsymbol{r}) + \boldsymbol{\Omega}_{D_b}(\boldsymbol{\Omega}\cdot\boldsymbol{r}) \tag{28}$$

or

$$\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_{iD_b} = [2\boldsymbol{\Omega}_0 + \boldsymbol{\Omega}]_{D_b}(\boldsymbol{\Omega}\cdot\boldsymbol{r}) \tag{29}$$

A. Level IAFA Case

When the IAFA rotation plane is tangent to the Earth surface, the vectors $\boldsymbol{\Omega}_0$, $\boldsymbol{\Omega}$, and $\boldsymbol{r}$, in body coordinates b are

$$\boldsymbol{\Omega}_{0b} = \begin{bmatrix} 0 \\ 0 \\ \Omega_0 \end{bmatrix} \tag{30a}$$

$$\boldsymbol{\Omega}_b = \begin{bmatrix} \Omega_N \cdot c\psi \\ -\Omega_N \cdot s\psi \\ \Omega_D \end{bmatrix} \tag{30b}$$

$$\boldsymbol{r}_b = \begin{bmatrix} r\cdot c\alpha \\ r\cdot s\alpha \\ 0 \end{bmatrix} \tag{30c}$$

where c denotes the cosine and s denotes the sine function, and where

$$\Omega_N = \Omega\cdot cL \tag{31a}$$

$$\Omega_D = -\Omega\cdot sL \tag{31b}$$

Therefore,

$$[2\boldsymbol{\Omega}_0 + \boldsymbol{\Omega}]_{D_b} = (2\Omega_0 + \Omega_D) \tag{32a}$$

and

$$\boldsymbol{\Omega}\cdot\boldsymbol{r} = \Omega_N\cdot r\cdot c\psi\cdot c\alpha - \Omega_N\cdot r\cdot s\psi\cdot s\alpha = \Omega_N\cdot r\cdot c(\psi+\alpha) \tag{32b}$$

Substitution of Eqs. (32) into Eq. (29) yields

$$\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_{iD_b} = (2\Omega_0 + \Omega_D)\Omega_N\cdot r\cdot c(\psi+\alpha) \tag{33}$$

Because $\alpha = \Omega_0 t$ and in view of Eq. (1) using the last equation we obtain the final result

$$f_{D_b} = (2\Omega_0 + \Omega_D)\Omega_N\cdot r\cdot c(\psi + \Omega_0 t) - g \tag{34}$$

Note that $\boldsymbol{f}_{D_b}$ reaches maximum when $\Omega_0 t = -\psi$. The pickoff sensor, at point pd, senses the passage of the accelerometer and emits a pulse when it occurs. The phase difference between the time of maximum signal to this time is the angle ψ. Note that the existence of g does not influence the determination of the maximum. In practice, it is more accurate to detect zero crossing than a maximum point of a sine or cosine signal; therefore, the phase measurement is between the zero crossing and the next pulse. The phase difference is then $\pi/2 + \psi$. Note that, again, g and any other constant signal added to the cosine signal have no influence on the determination of the mean crossing.

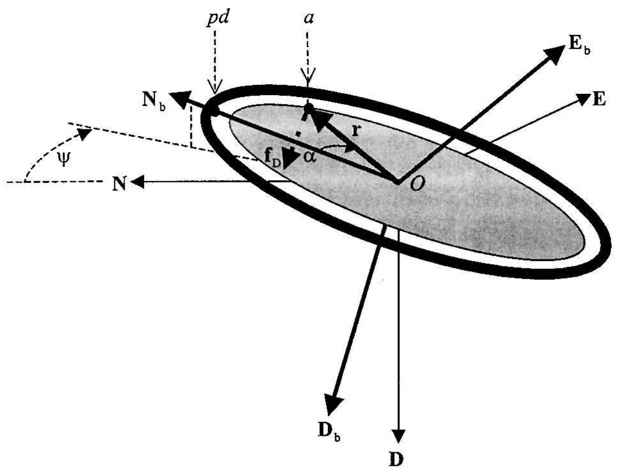

Fig. 3 Graphical description of the IAFA in a tilted position.

In principle, ψ can be found as just explained; however, as was mentioned before, in Sec. VI a more robust method will be given for extracting ψ from f_{D_b}, which uses the expression given in Eq. (34).

Finally, note that due to the relatively high rate of turn a relatively strong constant centripetal acceleration is acting on the accelerometer perpendicularly to its sensitive axis. Even when the accelerometer is well aligned, some of the centripetal acceleration may be sensed by the accelerometer. This acceleration, however, is constant; therefore, it does not affect the accuracy of the device. The only concern is a mechanical one, which has to be taken in account when considering the strength of the structure of the IAFA.

B. Tilted IAFA Case

In this case, which is shown in Fig. 3, the vehicle body coordinates are yawed, pitched, and rolled with respect to the l coordinates. As before, the yaw angle is denoted by ψ, and the pitch and roll angles are denoted by θ and ϕ, respectively. It is assumed that θ and ϕ are measured. The transformation matrix from l to b is

$$D_b^l = \begin{bmatrix} c\psi\cdot c\theta & s\psi\cdot c\theta & -s\theta \\ c\psi\cdot s\theta\cdot s\phi - s\psi\cdot c\phi & s\psi\cdot s\theta\cdot s\phi + c\psi\cdot c\phi & c\theta\cdot s\phi \\ c\psi\cdot s\theta\cdot c\phi + s\psi\cdot s\phi & s\psi\cdot s\theta\cdot c\phi - c\psi\cdot s\phi & c\theta\cdot c\phi \end{bmatrix} \tag{35}$$

Going back to Eq. (29), we realize that the vectors $\boldsymbol{\Omega}_0$ and $\boldsymbol{r}$, in body coordinates b, are still those given in Eqs. (30a) and (30c), but for $\boldsymbol{\Omega}_b$ we get now

$$\boldsymbol{\Omega}_b = D_b^l\boldsymbol{\Omega}_l \tag{36}$$

where

$$\boldsymbol{\Omega}_l^T = [\Omega_N \quad 0 \quad \Omega_D] \tag{37}$$

Equations (35–37) yield

$$\boldsymbol{\Omega}_b^T = \lfloor \Omega_{N_b} \quad \Omega_{E_b} \quad \Omega_{D_b} \rfloor \tag{38}$$

where

$$\Omega_{N_b} = \Omega_N\cdot c\psi\cdot c\theta - \Omega_D\cdot s\theta$$
$$\Omega_{E_b} = \Omega_N\cdot(c\psi\cdot s\theta\cdot s\phi - s\psi\cdot c\phi) + \Omega_D\cdot c\theta\cdot s\phi$$
$$\Omega_{D_b} = \Omega_N\cdot(c\psi\cdot s\theta\cdot c\phi + s\psi\cdot s\phi) + \Omega_D\cdot c\theta\cdot c\phi \tag{39}$$

Using Eqs. (30a) and (38), we obtain

$$[2\boldsymbol{\Omega}_0 + \boldsymbol{\Omega}]_{D_b} = \left(2\Omega_0 + \Omega_{D_b}\right) \tag{40}$$

Similarly, using Eqs. (30c) and (38), we obtain

$$\boldsymbol{\Omega}\cdot\boldsymbol{r} = \Omega_{N_b}\cdot r\cdot c\alpha + \Omega_{E_b}\cdot r\cdot s\alpha \tag{41}$$

Then using Eqs. (40) and (41) in Eq. (29) yields

$$\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_{iD_b} = \left(2\Omega_0 + \Omega_{D_b}\right) \cdot r \cdot \left[\Omega_{N_b} \cdot c\alpha + \Omega_{E_b} \cdot s\alpha\right] \tag{42}$$

Define the angle ζ as follows:

$$\zeta = \tan^{-1}\left(-\Omega_{E_b}/\Omega_{N_b}\right) \tag{43}$$

then Eq. (42) can be written as

$$\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_{iD_b} = \sqrt{\Omega_{N_b}^2 + \Omega_{E_b}^2} \cdot \left(2\Omega_0 + \Omega_{D_b}\right) \cdot r \cdot c(\zeta + \alpha) \tag{44}$$

In view of Eq. (1), we note that

$$\boldsymbol{f}_{D_b} = \left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_{iD_b} + g_{D_b} \tag{45}$$

Thus, from the last two equations and the relation $\alpha = \Omega_0 t$, we obtain

$$\boldsymbol{f}_{D_b} = \Omega_H \cdot \left(2\Omega_0 + \Omega_{D_b}\right) \cdot r \cdot c(\zeta + \Omega_0 t) - g_{b,D_b} \tag{46}$$

where Ω_H is the horizontal component of Earth rate in body defined by $\Omega_H = \sqrt{(\Omega_{N_b}^2 + \Omega_{E_b}^2)}$. Because $\boldsymbol{g}_1$ has only one component and it is in the D direction, then using Eq. (35) it is easy to see that

$$g_{b,D_b} = g \cdot c\theta \cdot c\phi \tag{47}$$

but because for a constant orientation g_{b,D_b} is constant, then as in the preceding case its value is immaterial.

Similar to ψ in the level case, ζ, too, is measurable. (Note that the existence of ψ in the expression for Ω_{D_b}, given in Eq. (39), is of no consequence because, as will be seen next, the value of the amplitude of $\boldsymbol{f}_{D_b}$ plays no role in the calculation of the azimuth.) We need to extract the correct azimuth angle ψ from ζ, Ω_{N_b}, and Ω_{D_b} and the connection between them. From Eq. (43) we get

$$\Omega_{N_b} \cdot \tan\zeta = -\Omega_{E_b} \tag{48}$$

which, using Eqs. (39), becomes

$$\begin{aligned}&(\Omega_N \cdot c\theta \cdot \tan\zeta + \Omega_N \cdot s\theta \cdot s\phi) \cdot c\psi \\ &\quad = \Omega_N \cdot c\phi \cdot s\psi - \Omega_D \cdot c\theta \cdot s\phi + \Omega_D \cdot s\theta \cdot \tan\zeta\end{aligned} \tag{49}$$

Using Eqs. (31), we can write the last equation as

$$\begin{aligned}&(c\theta \cdot \tan\zeta + s\theta \cdot s\phi) \cdot c\psi - c\phi \cdot s\psi \\ &\quad = \tan L \cdot (s\theta \cdot \tan\zeta - c\theta \cdot s\phi)\end{aligned} \tag{50}$$

Define

$$U = c\theta \cdot \tan\zeta + s\theta \cdot s\phi \tag{51a}$$

$$V = c\phi \tag{51b}$$

$$W = \tan L \cdot (s\theta \cdot \tan\zeta - c\theta \cdot s\phi) \tag{51c}$$

When the variables U, V, and W are used, Eq. (50) can be written as

$$U \cdot c\psi - V \cdot s\psi = W \tag{52}$$

or, alternatively, as

$$\sqrt{U^2 + V^2}\left[\left(U/\sqrt{U^2 + V^2}\right)c\psi - \left(V/\sqrt{U^2 + V^2}\right)s\psi\right] = W \tag{53}$$

Next define

$$K = W/\sqrt{U^2 + V^2} \tag{54a}$$

$$\mu = \tan^{-1}(V/U) \tag{54b}$$

Then Eq. (53) can be written as

$$(c\mu \cdot c\psi - s\mu \cdot s\psi) = K \tag{55a}$$

which yields

$$\cos(\mu + \psi) = K \tag{55b}$$

Consequently,

$$\psi = \cos^{-1}K - \mu \tag{56}$$

In this case of tilted IAFA, the apparatus yields the angle ζ rather than ψ. However, once ζ is known, and θ and ϕ are measured, we can compute ψ using Eqs. (51–56).

V. Azimuth Finding During Constant Velocity Motion

When the IAFA is installed on a vehicle that moves at a constant velocity, $\boldsymbol{\rho} \neq 0$. We assume that the horizontal components of the vehicle velocity in the l coordinate system are constant. As in the case of azimuth finding at rest, $\boldsymbol{\Omega}_0$ is still constant in the p system; therefore,

$$\left.\frac{\mathrm{d}\boldsymbol{\Omega}_0}{\mathrm{d}t}\right|_p = 0 \tag{57}$$

The body is still not rotating with respect to l, therefore, again

$$\boldsymbol{\omega}^* = 0 \tag{58}$$

Using Eqs. (57) and (58) in the general case expressed by Eq. (14) results in

$$\begin{aligned}\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_i &= \left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_p \times \boldsymbol{r} + \left.\frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t}\right|_p \times \boldsymbol{r} + (\boldsymbol{\Omega}_0 + \boldsymbol{\rho}) \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) \\ &\quad + (\boldsymbol{\Omega}_0 + \boldsymbol{\rho}) \times (\boldsymbol{\Omega} \times \boldsymbol{r}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \\ &\quad + (\boldsymbol{\Omega}_0 + \boldsymbol{\Omega} + \boldsymbol{\rho}) \times (\boldsymbol{\rho} \times \boldsymbol{r})\end{aligned} \tag{59}$$

Let us now examine each term on the right-hand side of Eq. (59). We can formally write

$$\left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_e = \left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_p + (\boldsymbol{\Omega}_0 + \boldsymbol{\rho} + \boldsymbol{\omega}^*) \times \boldsymbol{\Omega} \tag{60}$$

Because $\boldsymbol{\Omega}$ is constant in e,

$$\left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_e = 0 \tag{61}$$

When Eqs. (58) and (61) are used, Eq. (60) becomes

$$\left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_p = -\boldsymbol{\Omega}_0 \times \boldsymbol{\Omega} - \boldsymbol{\rho} \times \boldsymbol{\Omega} \tag{62}$$

Therefore,

$$\left.\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t}\right|_p \times \boldsymbol{r} = \boldsymbol{r} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{\Omega}) + \boldsymbol{r} \times (\boldsymbol{\rho} \times \boldsymbol{\Omega}) \tag{63}$$

We can also formally write

$$\left.\frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t}\right|_l = \left.\frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t}\right|_p + (\boldsymbol{\Omega}_0 + \boldsymbol{\rho}) \times \boldsymbol{\rho} \tag{64}$$

Now (see Ref. 5)

$$\underline{\rho}_l^T = \{V_E/(R_E + h) \quad -V_N/(R_N + h) \quad [V_E/(R_E + h)]\tan L\} \tag{65}$$

Because V_N and V_E are constants and because R_N and R_E, as well as L, change very slowly, it is quite accurate to assume that $\dot{\underline{\rho}}_l$ is negligible. This is still true even when h changes because the effect of this change on $\dot{\rho}_l$ is still negligible. Consequently,

$$\left.\frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t}\right|_l = 0 \tag{66}$$

Therefore, from Eq. (64) we obtain

$$\left.\frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t}\right|_p = -(\boldsymbol{\Omega}_0 + \boldsymbol{\rho}) \times \boldsymbol{\rho} \tag{67}$$

Therefore,

$$\left.\frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t}\right|_p \times \boldsymbol{r} = -[(\boldsymbol{\Omega}_0 + \boldsymbol{\rho}) \times \boldsymbol{\rho}] \times \boldsymbol{r} = \boldsymbol{r} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{\rho}) + \boldsymbol{r} \times (\boldsymbol{\rho} \times \boldsymbol{\rho}) \tag{68}$$

Substitution of Eqs. (63) and (68) into Eq. (59) yields

$$\begin{aligned}\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_i &= [\boldsymbol{r} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{\Omega}) + \boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) + \boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \\ &\quad + \boldsymbol{\Omega} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})] + [\boldsymbol{r} \times (\boldsymbol{\rho} \times \boldsymbol{\Omega}) \\ &\quad + \boldsymbol{r} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{\rho}) + \boldsymbol{r} \times (\boldsymbol{\rho} \times \boldsymbol{\rho}) + \boldsymbol{\rho} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) \\ &\quad + \boldsymbol{\rho} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) + (\boldsymbol{\Omega}_0 + \boldsymbol{\Omega} + \boldsymbol{\rho}) \times (\boldsymbol{\rho} \times \boldsymbol{r})]\end{aligned} \tag{69}$$

The first five terms on the right-hand side of Eq. (69) are identical to the nominal acceleration given in Eq. (23). Let us denote them by $\boldsymbol{a}$, that is,

$$\begin{aligned}\boldsymbol{a} &= \boldsymbol{r} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{\Omega}) + \boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) + \boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \\ &\quad + \boldsymbol{\Omega} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r})\end{aligned} \tag{70}$$

Let us denote the rest of the terms on the right-hand side of Eq. (69) by $\boldsymbol{b}$, then noting that $\boldsymbol{\rho} \times \boldsymbol{\rho} = 0$, we obtain

$$\begin{aligned}\boldsymbol{b} &= \boldsymbol{r} \times (\boldsymbol{\rho} \times \boldsymbol{\Omega}) + \boldsymbol{r} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{\rho}) + \boldsymbol{\rho} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) \\ &\quad + \boldsymbol{\rho} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) + (\boldsymbol{\Omega}_0 + \boldsymbol{\Omega} + \boldsymbol{\rho}) \times (\boldsymbol{\rho} \times \boldsymbol{r})\end{aligned} \tag{71}$$

It was shown before [see the passage from Eq. (23) to Eq. (29)] that, in body axes, the down component of $\boldsymbol{a}$ is given by

$$\underline{\boldsymbol{a}}_{b,D_b} = [2\boldsymbol{\Omega}_0 + \boldsymbol{\Omega}]_{b,D_b}(\boldsymbol{\Omega} \cdot \boldsymbol{r}) \tag{72}$$

To compute the body down component of $\boldsymbol{b}$, we turn again to Eq. (24):

$$\boldsymbol{c} \times (\boldsymbol{m} \times \boldsymbol{n}) = \boldsymbol{m}(\boldsymbol{c} \cdot \boldsymbol{n}) - \boldsymbol{n}(\boldsymbol{c} \cdot \boldsymbol{m}) \tag{73}$$

and apply it to the terms that comprise $\boldsymbol{b}$; thus,

$$\boldsymbol{r} \times (\boldsymbol{\rho} \times \boldsymbol{\Omega}) = \boldsymbol{\rho}(\boldsymbol{r} \cdot \boldsymbol{\Omega}) - \boldsymbol{\Omega}(\boldsymbol{r} \cdot \boldsymbol{\rho}) \tag{74a}$$

$$\boldsymbol{r} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{\rho}) = \boldsymbol{\Omega}_0(\boldsymbol{r} \cdot \boldsymbol{\rho}) - \boldsymbol{\rho}(\boldsymbol{r} \cdot \boldsymbol{\Omega}_0) \tag{74b}$$

$$\boldsymbol{\rho} \times (\boldsymbol{\Omega}_0 \times \boldsymbol{r}) = \boldsymbol{\Omega}_0(\boldsymbol{\rho} \cdot \boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{\rho} \cdot \boldsymbol{\Omega}_0) \tag{74c}$$

$$\boldsymbol{\rho} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{\Omega}(\boldsymbol{\rho} \cdot \boldsymbol{r}) - \boldsymbol{r}(\boldsymbol{\rho} \cdot \boldsymbol{\Omega}) \tag{74d}$$

$$\begin{aligned}(\boldsymbol{\Omega}_0 + \boldsymbol{\Omega} + \boldsymbol{\rho}) \times (\boldsymbol{\rho} \times \boldsymbol{r}) &= \boldsymbol{\rho}[(\boldsymbol{\Omega}_0 + \boldsymbol{\Omega} + \boldsymbol{\rho}) \cdot \boldsymbol{r}] \\ &\quad - \boldsymbol{r}[(\boldsymbol{\Omega}_0 + \boldsymbol{\Omega} + \boldsymbol{\rho}) \cdot \boldsymbol{\rho}]\end{aligned} \tag{74e}$$

Because, as we noted before, $\boldsymbol{r}$ and $\boldsymbol{\Omega}_0$ are perpendicular to one another, then $\boldsymbol{r} \cdot \boldsymbol{\Omega}_0 = 0$. When this is used in Eqs. (74), and the results are substituted into Eq. (71), we obtain

$$\begin{aligned}\boldsymbol{b} &= \boldsymbol{\rho}[2(\boldsymbol{r} \cdot \boldsymbol{\Omega}) + (\boldsymbol{\rho} \cdot \boldsymbol{r})] + 2\boldsymbol{\Omega}_0(\boldsymbol{r} \cdot \boldsymbol{\rho}) \\ &\quad - \boldsymbol{r}[2(\boldsymbol{\rho} \cdot \boldsymbol{\Omega}_0) + 2(\boldsymbol{\rho} \cdot \boldsymbol{\Omega}) - (\boldsymbol{\rho} \cdot \boldsymbol{\rho})]\end{aligned} \tag{75}$$

The accelerometer measures only the component of $\boldsymbol{b}$ that is in the direction of the body down axis. Because $\boldsymbol{r}$ is perpendicular to the body down axis, the part of $\boldsymbol{b}$ that is in the direction of $\boldsymbol{r}$ vanishes. Adding the remaining parts of $\boldsymbol{b}$ to $\boldsymbol{a}$ we obtain from Eq. (69)

$$\left.\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right|_{iD_b} = [2\boldsymbol{\Omega}_0 + \boldsymbol{\Omega} + 2\boldsymbol{\rho}]_{D_b}(\boldsymbol{\Omega} \cdot \boldsymbol{r}) + [\boldsymbol{\rho} + 2\boldsymbol{\Omega}_0]_{D_b}(\boldsymbol{\rho} \cdot \boldsymbol{r}) \tag{76}$$

From the last equation and using Eqs. (1) and (30a), we obtain

$$\boldsymbol{f}_{D_b} = 2\Omega_0[(\boldsymbol{\Omega} + \boldsymbol{\rho}) \cdot \boldsymbol{r}] + (\boldsymbol{\Omega} + 2\boldsymbol{\rho})_{b,D_b}(\boldsymbol{\Omega} \cdot \boldsymbol{r}) + \boldsymbol{\rho}_{b,D_b}(\boldsymbol{\rho} \cdot \boldsymbol{r}) - g_{D_b} \tag{77}$$

Note that due to the huge difference between the value of Ω_0 on one hand and that of $\boldsymbol{\Omega}$ and $\boldsymbol{\rho}$ on the other hand, the term $2\Omega_0[(\boldsymbol{\Omega} + \boldsymbol{\rho}) \cdot \boldsymbol{r}]$ is, by far, the dominant term in the expression for $\boldsymbol{f}_{D_b}$. Unlike the preceding case of motionless IAFA, where we treated the level and tilted IAFA separately, here we consider only the tilted IAFA case, and the level IAFA case is derived directly from the latter by letting the pitch and roll angles be zero.

To evaluate the terms in Eq. (77), we need to express the vectors in some coordinate system. The simplest is to express all vectors in the IAFA body coordinates. The value of $\boldsymbol{\Omega}_b$ is given in Eqs. (39). To evaluate $\boldsymbol{\rho}_b$ we use

$$\boldsymbol{\rho}_b = D_b^l\, \boldsymbol{\rho}_l \tag{78}$$

Because near Earth $R_E + h \cong R_N + h \cong R$, and because

$$V_N = V \cdot cH \tag{79a}$$

$$V_E = V \cdot sH \tag{79b}$$

we can write Eq. (65) as

$$\boldsymbol{\rho}_l^T = [sH \quad -cH \quad sH \cdot \tan L](V/R) \tag{80}$$

We distinguish between the torqued case and the case of no velocity information.

A. Torqued Case

We found [see Eq. (41)] that

$$\boldsymbol{\Omega} \cdot \boldsymbol{r} = \Omega_{N_b} \cdot r \cdot c\alpha + \Omega_{E_b} \cdot r \cdot s\alpha \tag{81}$$

where Ω_{N_b} and Ω_{E_b} are given in Eqs. (39). We use Eq. (30c) to write

$$\boldsymbol{\rho} \cdot \boldsymbol{r} = \rho_{N_b} \cdot r \cdot c\alpha + \rho_{E_b} \cdot r \cdot c\alpha \tag{82}$$

and we need to evaluate the components of $\boldsymbol{\rho}_b$. If the vehicle on which the IAFA is installed is torqued to maintain the local-level north pointing (l coordinates) orientation, then V_N and V_E cause the IAFA to stay at constant yaw pitch and roll angles with respect to the l coordinate system. Using Eqs. (35), (78), and (80), we obtain the following expressions for the components of $\boldsymbol{\rho}_b$:

$$\begin{aligned}\rho_{N_b} &= [(c\psi \cdot c\theta)s(H) - (s\psi \cdot c\theta) \cdot c(H) \\ &\quad - s\theta \cdot s(H) \cdot \tan L](V/R)\end{aligned} \tag{83a}$$

$$\begin{aligned}\rho_{E_b} &= [(c\psi \cdot s\theta \cdot s\phi - s\psi \cdot c\phi) \cdot sH - (s\psi \cdot s\theta \cdot s\phi \\ &\quad + c\psi \cdot c\phi) \cdot cH + c\theta \cdot s\phi \cdot sH \cdot \tan L](V/R)\end{aligned} \tag{83b}$$

$$\begin{aligned}\rho_{D_b} &= [(c\psi \cdot s\theta \cdot c\phi + s\psi \cdot s\phi) \cdot sH - (s\psi \cdot s\theta \cdot c\phi \\ &\quad - c\psi \cdot s\phi) \cdot cH + c\theta \cdot c\phi \cdot sH \cdot \tan L](V/R)\end{aligned} \tag{83c}$$

We can now write the value of $[\boldsymbol{\Omega} + 2\boldsymbol{\rho}]_{D_b}$ as follows:

$$[\boldsymbol{\Omega} + 2\boldsymbol{\rho}]_{D_b} = \Omega_{D_b} + 2\rho_{D_b} \tag{84}$$

Substitution of Eqs. (81), (82), (84), and (30c) into Eq. (77) yields

$$\begin{aligned}\boldsymbol{f}_{D_b} &= \big[2\Omega_0\big(\Omega_{N_b} + \rho_{N_b}\big) \cdot r + \big(\Omega_{D_b} + 2\rho_{b,D_b}\big)\Omega_{N_b} \cdot r \\ &\quad + \rho_{D_b}\rho_{N_b} \cdot r\big] \cdot c\alpha + \big[2\Omega_0\big(\Omega_{E_b} + \rho_{E_b}\big) \cdot r \\ &\quad + \big(\Omega_{D_b} + 2\rho_{b,D_b}\big)\Omega_{E_b} \cdot r + \rho_{D_b}\rho_{E_b} \cdot r\big] \cdot s\alpha - g_{D_b}\end{aligned} \tag{85}$$

Define

$$A = 2\Omega_0\left(\Omega_{N_b} + \rho_{N_b}\right) \cdot r + \left(\Omega_{D_b} + 2\rho_{b,D_b}\right)\Omega_{N_b} \cdot r + \rho_{D_b}\rho_{N_b} \cdot r \tag{86a}$$

$$B = 2\Omega_0\left(\Omega_{E_b} + \rho_{E_b}\right) \cdot r + \left(\Omega_{D_b} + 2\rho_{b,D_b}\right)\Omega_{E_b} \cdot r + \rho_{D_b}\rho_{E_b} \cdot r \tag{86b}$$

then Eq. (85) can be written as

$$\boldsymbol{f}_{D_b} = A \cdot c\alpha + B \cdot s\alpha = \sqrt{A^2 + B^2} \times \left[\left(A\big/\sqrt{A^2 + B^2}\right)c\alpha + \left(B\big/\sqrt{A^2 + B^2}\right)s\alpha\right] - g_{D_b} \tag{87}$$

Let

$$\xi = \tan^{-1}(-B/A) \tag{88}$$

then, Eq. (85) can be written as

$$f_{D_b} = \sqrt{A^2 + B^2} \cdot c(\xi + \alpha) - g_{b,D_b} \tag{89}$$

where g_{b,D_b} is given in Eq. (47). As in the preceding cases, the value of ξ is found from measurements and computations, as will be explained in the following section.

Note that in the special case when the IAFA is stationary, $\boldsymbol{\rho}_b = 0$, and ξ of Eq. (88) is identical to ζ of Eq. (43), which was developed directly for this case. Similarly, $\boldsymbol{f}_{D_b}$ of Eq. (89) is identical to that of Eq. (46).

With ξ on hand, we can find ψ as follows. Using Eqs. (86) and (88), we obtain

$$\left[\left(2\Omega_0 + \Omega_{D_b} + 2\rho_{D_b}\right)\Omega_{N_b} + \left(\rho_{D_b} + 2\Omega_0\right)\rho_{N_b}\right]\tan\xi - \left(2\Omega_0 + \Omega_{D_b} + 2\rho_{D_b}\right)\Omega_{E_b} + \left(\rho_{D_b} + 2\Omega_0\right)\rho_{E_b} = 0 \tag{90}$$

This is a nonlinear equation in ψ because Ω_{D_b}, Ω_{N_b}, ρ_{D_b}, ρ_{N_b}, Ω_{E_b}, and ρ_{E_b} are (known) nonlinear functions of ψ. This equation can be solved numerically using some iterative algorithm, for example, the Newton–Raphson algorithm, to yield ψ.

B. Case of No Velocity Information

When no information of the vehicle velocity is available, then for the duration of the azimuth finding process we do not slew the platform on which the IAFA is placed. The result of this inaction is a growth of ψ, θ, and ϕ. However, because θ and ϕ are monitored, our only concern is the growth of ψ, the angle we are trying to find. For example, if we fly due east at a velocity of 330 m/s at a 45° latitude, the change of ψ is about 10 arcsec/s, much of which we can compensate. In conclusion, we can cease slewing the vehicle by $\boldsymbol{\rho}$ and still obtain an accurate measurement of the azimuth using the IAFA.

Finally, when the IAFA is moving at a constant velocity motion and the body axes are level, then this can be handled as a special case of the preceding case where θ and ϕ are set to zero [see Eqs. (39), (47), and (83)].

VI. Robust Method for Extracting the Angular Information

In the preceding sections we mentioned an approach for extracting ψ (or ζ) from the signal $\boldsymbol{f}_{D_b}$ measured by the accelerometer at point a. It was based on the measurement of the time difference between the mean crossing of the signal and the time when the photodetector at point pd emitted a pulse. A method that is more robust and accurate is presented next. In all cases described earlier, we needed to extract the phase from $\boldsymbol{f}_{D_b}$ where the latter was given in the form

$$f_{D_b} = A \cdot c(\Omega_0 t + \eta) + a' \tag{91}$$

and where η is the phase angle and a' is a constant gravity acceleration. To find η, we first compute an integral ν_s defined as follows:

$$\nu_s = \int_{t=0}^{t=2\pi/\Omega_0} f_{D_b} \cdot s(\Omega_0 t)\, \mathrm{d}t \tag{92}$$

Substituting the expression for $\boldsymbol{f}_{D_b}$ given in Eq. (91) into the last equation yields

$$\nu_s = \int_{t=0}^{t=2\pi/\Omega_0} [A \cdot c(\Omega_0 t + \eta) + s] \cdot s(\Omega_0 t)\, \mathrm{d}t \tag{93}$$

Expending the last expression, we obtain

$$\nu_s = A \cdot \int_{t=0}^{t=2\pi/\Omega_0} c(\Omega_0 t + \eta) \cdot s(\Omega_0 t)\, \mathrm{d}t + a' \cdot \int_{t=0}^{t=2\pi/\Omega_0} s(\Omega_0 t)\, \mathrm{d}t \tag{94}$$

The first integral is equal to $\frac{1}{2}s\psi$ and the second integral vanishes; thus,

$$\nu_s = -A \cdot \tfrac{1}{2}s\eta \tag{95}$$

Similarly, compute ν_c defined as follows:

$$\nu_c = \int_{t=0}^{t=2\pi/\Omega_0} \boldsymbol{f}_{D_b} \cdot c(\Omega_0 t)\, \mathrm{d}t \tag{96}$$

Following the steps that led to the expression given in Eq. (95), it can be shown that

$$\nu_c = A \cdot \tfrac{1}{2}c\eta \tag{97}$$

Finally, from Eqs. (95) and (97), we obtain

$$\eta = -\tan^{-1}(\nu_s/\nu_c) \tag{98}$$

We can use this computational procedure to compute ψ, ζ, and ξ defined in Eqs. (34), (46), and (89), respectively.

VII. Experiment Results

Figure 4 presents the experimental setup used to verify and test the IAFA concept. The setup was a breadboard model that was installed on a dividing table, which was placed on a granite table. The inner platform, on which the accelerometer was mounted, was driven by an electric precision low caging direct drive brushless motor of the kind used to drive video cassette recorders. The accelerometer was a servo-type accelerometer. The manufacturer published specifications were as follows. Its uncompensated bias was specified to be less than 8 mg, and its one year bias stability to be less than 1200 μg. Its uncompensated scale factor was stated to be less than 10%, and its scale factor one year stability to be less than 1200 ppm. Its specified resolution was better than 1 μg and its input axis misalignment better than 2 mrad. Actually, two accelerometers were used to eliminate constant accelerations, the gravity acceleration influence, and any other external undesired inputs that affect the accelerometers in the same manner. The accelerometers were put on one diameter at a distance of 3 cm from the center of the IAFA axis. Their outputs were differenced, which, as mentioned, eliminated the common disturbance acceleration while adding the useful measurement. This addition of the useful measurement resulted because at their

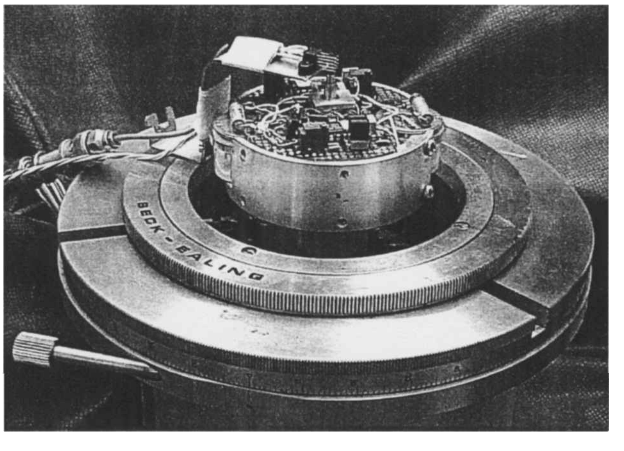

Fig. 4 Laboratory model of the experimental IAFA.

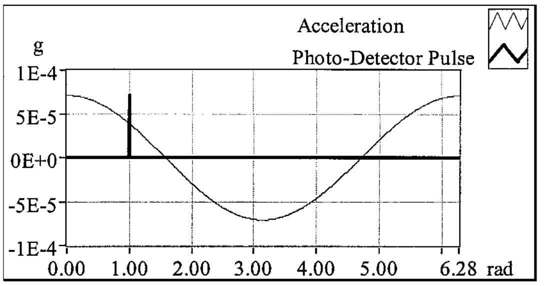

Fig. 5 Typical accelerometer and photodetector outputs.

positions they both read the same cosine signal but with opposite phases.

The signal of each accelerometer was sampled at a rate of 1000 samples per period (turn). The inner platform rotated at a rate of 25 revolutions per second; therefore, there were 25 periods per second. Consequently, the temporal sampling rate of the accelerometer signals was 25,000 samples per second. The sampled signal passed through a signal-conditioning block and then through a V to F converter. The distance between two consecutive pulses emitted by the photodiode (Fig. 2) determined the actual period. Therefore low-frequency speed changes that might have occurred during the experiment did not influence the results.

Before processing the data to extract the angular information described in Sec. VI, data were collected over a 5-s interval. During that time interval, the platform completed 125 periods. The data of the 125 periods were averaged to yield an average cosine shaped function. Next an analytic cosine shaped curve was fitted to the latter using a minimum least-squares fit algorithm. The resulting analytical function was used in extracting the angular information using the algorithm described in Sec. VI. A typical sample of the analytic signal is presented in Fig. 5, where the photodetector emitted signal is also seen. The value of the cosine function is presented as a function of the rotation angle in radians. North is detected at 0.00°, and the body north axis is where the photodetector is. An inspection reveals that the body north is at about 1 rad from north. In other words, the measured azimuth angle is 57.3°. The IAFA system computes this value exactly using the robust algorithm presented in Sec. VI. With this kind of data processing, low-frequency as well as high-frequency variations of the accelerometer bias and scale factor are reduced to minimum. Because of the centrifugal acceleration, accelerometer constant misalignment introduces high-acceleration value, but because it is constant, it is nonconsequential.

Figure 6 presents test results of the IAFA at three azimuth angles; namely, 0, 30, and 60°. In the examples presented in Fig. 6, at each azimuth angle the data was averaged over 1 min rather than 5 s as usual, and the process of azimuth finding was repeated 10 times. All 10 results are presented in Fig. 6 for each azimuth angle. For example, when the body was placed at zero azimuth angle (Fig. 6a), the first IAFA measured azimuth was a little over 0.05° to the west; at the second trial it was a little over 0.05° east, and so on. The average error over the 10 trials for this azimuth was 0.0242° west. When the body was placed at 30° azimuth (Fig. 6b), this value was 0.0866° west, and when the body was placed at an azimuth of 60° (Fig. 6c), the average error was 0.0360° west. Note that a part of the measured error was the error in the placement of the test equipment.

The IAFA needs no warm up or initial alignment. Random low-frequency variations in the rotation rate of the IAFA platform, if they occur, are noninfluential because the period is determined each cycle by the time interval between two consecutive photodetector pulses. The averaging and analytic-curve fitting that are employed in the signal processing stage also reduce the random high-frequency disturbances of all sources to a minimum. Finally, it is anticipated that with a better fabricated apparatus the error can be brought down to the 0.01° level.

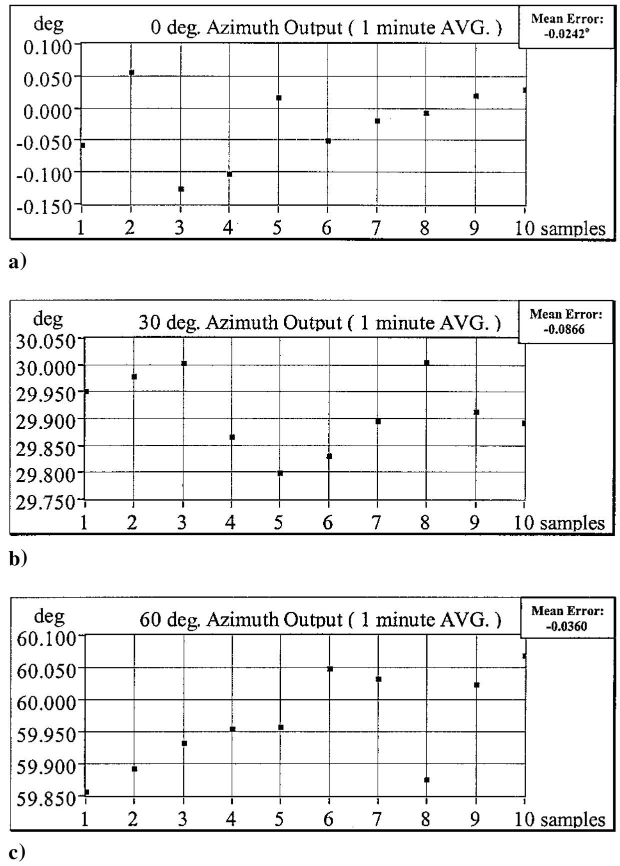

Fig. 6 IAFA azimuth determination results for a) 0, b) 30, and c) 60° azimuth angles.

VIII. Conclusions

This work presented a new inertial apparatus for finding the azimuth of a platform. A detailed description of the apparatus for azimuth finding, which is based on the measurements of an accelerometer that is placed vertically on a rotating plate, was given. The measured acceleration is a Coriolis acceleration generated when the accelerometer travels on a circular path. Formulas for extracting the azimuth in the case where the apparatus is level, as well when it is not level, were developed. A robust algorithm was presented for extracting the azimuth information from the accelerometer measurements. Results of an experimental setup were presented and discussed.

The main advantages of the new apparatus are as follows. A relatively accurate azimuth determination is accomplished very fast, in a fraction of a minute. It is insensitive to the accelerometer bias or scale factor as long as they stay constant during the rotation cycle, and they normally do. It is insensitive to accelerometer calibration and to the accuracy in acceleration magnitude reading; what really matters is the accelerometer resolution. It is insensitive to the exact value of the turning speed as long as it stays constant during the turn. It has low sensitivity to errors in the knowledge of pitch and roll as long as the pitch and roll angles themselves are reasonable. The apparatus is light, small, and consumes little power. Finally, the device can be used for azimuth finding in terrestrial missions as well as in orbiting spacecraft.

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